11/1/23 Feigin–Frenkel Isomorphism

Speaker: Kirill
Notes: Nikolay

$\mathfrak{g}$ - semisimple Lie alg $/\mathbb{C}$

Fix κ - non degen inv. bilinear form on $\mathfrak{g}$

$\hat{\mathfrak{g}}_\kappa$ - affine Lie alg at level κ

Def 1) Vacuum module $\mathbb{V}_\kappa(\mathfrak{g}) := \mathcal{U}\hat{\mathfrak{g}}_\kappa \otimes_{\mathfrak{g}[[t]] \oplus \mathbb{C}\cdot 1} \mathbb{C}$

2) $\mathfrak{z}_\kappa(\mathfrak{g}) := \mathrm{End}_{\hat{\mathfrak{g}}_\kappa}(\mathbb{V}_\kappa(\mathfrak{g})) = \mathbb{V}_\kappa(\mathfrak{g})^{\mathfrak{g}[[t]]}$

Fact: is commutative algebra

Thm (Feigin–Frenkel)

$$\mathfrak{z}_\kappa(\mathfrak{g}) = \begin{cases} \mathbb{C} & \kappa \neq \kappa_{crit} \\ \mathbb{C}[S_{n,(m)}]_{\substack{n \in \text{exps of } \mathfrak{g} \\ m \leq -2}} & \end{cases} \qquad (*)$$

Recall HC: $Z(\mathcal{U}\mathfrak{g}) \simeq \mathbb{C}[\mathfrak{h}^*]^W = \mathbb{C}[S_n]_{n \in \text{exps of } \mathfrak{g}}$.

Note: $(*)$ is not canonical: depends on a choice of coord on formal disk $\mathbb{D}$.

Consider $\mathfrak{g} = \mathfrak{sl}_2$. Then $\mathfrak{z}_{\kappa_{crit}}(\mathfrak{sl}_2) = \mathbb{C}[S_{(m)}]_{m \leq -2}$.

$$\Rightarrow S(z) = \sum_{m \in \mathbb{Z}} S_{(m)} z^{-m-2}$$

Segal–Sugawara operator.

Change of coord: $z \mapsto \varphi(z)$.

$$S(\varphi(z)) = S(z) \cdot \varphi'(z)^2 - \frac{1}{2}\{\varphi, S\}$$

Schwarzian derivative.

Projective connection: $\mathbb{D} = \mathrm{Spec}\, \mathbb{C}[[t]]$, $\Omega^1_{\mathbb{D}} \simeq \mathbb{C}[[t]]\, dt$

$\Omega_{\mathbb{D}}^{1/2}$, $\Omega_{\mathbb{D}}^{-1/2}$ line bundles are well-defined.

Def A <u>projective connection</u> on $\mathbb{D}$ is a 2^{nd} order diff op

$\rho : \Omega_{\mathbb{D}}^{-1/2} \longrightarrow \Omega_{\mathbb{D}}^{1/2}$ of form $t \longmapsto \partial_t^2 - v(t)$.

Fact: $v(t)$ transforms under coordinate changes the same as $S(z)$.

Can consider: $S_{(m)} : \left\{ \begin{matrix} \text{projective} \\ \text{connections} \end{matrix} \right\} \longrightarrow \mathbb{C}$.

$$\partial_t^2 - v(t) \longmapsto t^{-m-2}\text{-coeff of } v(t)$$

Let $Op_{PGL_2}(\mathbb{D})$ = moduli space of projective connections

Thm $\operatorname{Spec} \mathfrak{z}_{\kappa_{crit}}(sl_2) \xrightarrow{\sim} Op_{PGL_2}(\mathbb{D})$, which is $Aut(\mathbb{D})$-equivariant.

Idea Replace proj. connection with some lin. alg. data.

Given proj. connection $\partial_t^2 - v(t)$, can consider a flat connection on $\mathcal{O}_{\mathbb{D}}^2$ (= triv. rk. 2 v.b on $\mathbb{D}$).

$$\nabla = \partial_t + \begin{pmatrix} 0 & v(t) \\ 1 & 0 \end{pmatrix}.$$

Equivalently, it's the data of a v.b. of rk. 2 w/ flat connection $(\mathcal{F}_0, \nabla)$ & a line sub-bundle $\mathcal{F}_1 \subseteq \mathcal{F}_0$ s.t

$$\mathcal{F}_1 \longrightarrow \mathcal{F}_0 \otimes \Omega_{\mathbb{D}}^1 \longrightarrow \mathcal{F}_0/\mathcal{F}_1 \otimes \Omega_{\mathbb{D}}^1$$ composition is isom.

To make it canonical, need to consider $(\mathcal{F}_0, \mathcal{F}_1, \nabla)$, where

$\mathcal{F}_0$ = principal PGL_2-bundle on $\mathbb{D}$.

∇ = connection on $\mathcal{F}_0$

$\mathcal{F}_1 \subseteq \mathcal{F}_0$, B-reduction of $\mathcal{F}_0$.

nowhere vanishing section of $\mathbb{P}^1 \overset{PGL_2}{\times} \mathcal{F}_0$.

§ Opers

G - reductive alg. gp, $\mathfrak{g} = \mathrm{Lie}\, G$.

B - fixed Borel,

$N = [B, B]$.

In $\mathfrak{g}/\mathfrak{b}$, there is a locally-closed B-orbit s.t it's invariant under N & negative simple root components are $\neq 0$, denoted $\underline{\mathbb{O}}$

Given scheme X, G-bundle P_G, can form associated bundle
$$\begin{array}{c} P_G \\ \downarrow \\ X \end{array}$$

$\mathbb{O} \subseteq \mathfrak{g}/\mathfrak{b} \rightsquigarrow \mathbb{O}_{P_G} \subseteq (\mathfrak{g}/\mathfrak{b})_{P_G}$.

<u>Def</u> A <u>G-oper on X</u> (X = smooth scheme) is triple (P_G, P_B, ∇):

- P_G is G-bundle on X
- P_B is B-reduction of P_G
- ∇ - connection on P_G s.t $\mathrm{Im}(\nabla)$ in $(\mathfrak{g}/\mathfrak{b})_{P_G} \otimes \Omega^1_X$ is inside $\mathbb{O}_{P_G} \otimes \Omega^1_X$.

<u>Note</u> We take $X = \mathbb{D}$.

<u>Ex</u> 1) $G = GL_n$. An oper is equivalent to data:

- $(\mathcal{E}_0, \nabla)$ - rank n v.b w/ connection ∇
- Full flag $0 = \mathcal{E}_n \subset \mathcal{E}_{n-1} \subset \cdots \subset \mathcal{E}_1 \subseteq \mathcal{E}_0$

s.t $\nabla : \mathcal{E}_i \longrightarrow \mathcal{E}_{i+1} \otimes \Omega^1_{\mathbb{D}}$ (Griffiths Transversality) &

$\overline{\nabla} : gr_i \mathcal{E} \xrightarrow{\sim} gr_{i+1} \mathcal{E} \otimes \Omega^1_{\mathbb{D}}$.

Ex 2) G - simple of adjoint type.

$\{F_\alpha\}$ - negative simple roots.

In local coord. t, such a connection ∇ looks like

$$\nabla = \partial_t + \sum_\alpha \varphi_\alpha(t) \cdot F_\alpha + \eta, \quad \eta \in \Omega^1_{\mathbb{D}} \otimes \mathfrak{b}$$

$\varphi_\alpha(t) \in \Omega^1_{\mathbb{D}}$ s.t. $\varphi_\alpha(t) \neq 0$ $\forall t$.

$\check{G}$ = Langlands dual of G.

$\mathrm{Aut} = \mathrm{Aut}(\mathbb{D})$

$\mathrm{Aut}_* = \mathrm{Aut}(\mathbb{D}, 0)$

Thm (Feigin-Frenkel, BD)

There is Aut - equivariant equivalence

$$\mathrm{Spec}(\mathfrak{Z}_{\kappa_{crit}}(\mathfrak{g})) \xrightarrow{\sim} \mathrm{Op}_{\check{G}}(\mathbb{D}).$$

Additional compatibility is geometric Satake:

$G(\mathcal{O}) \curvearrowright \mathrm{Gr}_G$ = affine - Grassmannian

$\text{D-mod}(\mathrm{Gr}_G)^{G(\mathcal{O})}$: is symmetric monoidal abelian cat.

Thm (Geom. Satake) There is sym. monoidal equiv.

$$\mathcal{F} : \mathrm{Rep}_{\check{G}} \xrightarrow{\sim} \text{D-mod}(\mathrm{Gr}_G)^{G(\mathcal{O})}.$$

Consider: $\Gamma_{crit} : \text{D-mod}(\mathrm{Gr}_G)^{G(\mathcal{O})} \longrightarrow \hat{\mathfrak{g}}_{\kappa_{crit}}\text{-rep}_{reg,\, G(\mathcal{O})\text{-int}}$

$$\mathcal{M} \longmapsto \Gamma(\mathcal{M} \otimes \underbrace{\mathcal{L}_{crit}}_{\det^{-1/2}})$$

$V \in \mathrm{Rep}_{G^\vee} \rightsquigarrow V \times^{G^\vee} P_{G^\vee}$ on $\mathrm{Op}_{G^\vee}(\mathbb{D})$

where $P_{G^\vee} \rightsquigarrow \mathrm{Op}_{G^\vee}(\mathbb{D})$ is tautological $G^\vee$-bundle.

Compatibility of F.F. w/ geometric Satake.

Thm $\Gamma_{crit}(\mathrm{Op}_{G^\vee}(\mathbb{D}), \mathcal{F}(V)) \xrightarrow{\sim} V \otimes_{\mathfrak{Z}} \Gamma(\mathrm{Op}_{G^\vee}(\mathbb{D}), V \times^{\check{G}} P_{G^\vee})$

Satake

Break

Construction of FF-map: (aka "Birth of oper")

Want Aut-equiv map $\mathrm{Spec}(\mathfrak{Z}_{crit}(\mathfrak{g})) \xrightarrow{\sim} \mathrm{Op}_{\check{G}}(\mathbb{D})$

$\Longleftrightarrow$ Giving Aut-equiv. family of opers on

$\mathrm{Spec}(\mathfrak{Z}_{\kappa_{crit}}(\mathfrak{g})) \times \mathbb{D}$.

Let $\mathfrak{Z} = \mathfrak{Z}_{\kappa_{crit}}(\mathfrak{g})$.

Strategy: 1) Construct Aut-equiv. $\check{G}$-bundle on $\mathrm{Spec}(\mathfrak{Z})$ & a Aut_*-equiv. reduction to $\check{B} \subset \check{G}$.

2) Construct a family of opers on $\mathrm{Spec}\,\mathfrak{Z} \times \mathbb{D}$ using Aut-equivariance.

How to construct $\check{G}$-bundle?

Tannakian formalism!

$\left(\text{Giving principal } G^\vee\text{-bundle on } \mathrm{Spec}\,\mathfrak{Z}\right) \longleftrightarrow$ Giving exact sym. monoidal functor $P: \mathrm{Rep}^{f.d}_{G^\vee} \to \mathfrak{Z}\text{-mod}^{f.g\ proj}$

$\begin{matrix} P_{G^\vee} \\ \downarrow \\ \mathrm{Spec}\,\mathfrak{Z} \end{matrix} \rightsquigarrow P(V) = V \times^{\check{G}} P_{G^\vee}$ associated bundle.

Define a functor $P^{o}_{G^\vee} : \mathrm{Rep}^{fd}(G^\vee) \longrightarrow \mathfrak{Z}\text{-mod}^{fg, proj}$

$$V \longmapsto \Gamma_{crit}(Gr_G, \mathcal{F}_V)^{G(O)}$$

<u>Thm</u> (Raskin) 1) $P^o_{G^\vee}(V)$ is fin. gen'd projective.

2) $H^i_{crit}(Gr_G, \mathcal{F}_V) = 0 \ \forall i > 0$.

3) $P^o_{G^\vee}$ is exact.

<u>Prop</u> $P^o_{G^\vee}$ is symm. monoidal (BD).

$\rightsquigarrow$ $\check{G}$-bundle on $\mathrm{Spec}(\mathfrak{Z}_\chi)$, Aut-equivariant.

§ $B^\vee$-reduction

<u>Plucker construction:</u> $G = G^\vee$, $B = B^\vee$

<u>Prop</u> Given a G-bundle P_G on an affine scheme $\mathrm{Spec}\, A$,

$$\left\{ \begin{matrix} B\text{-reduction} \\ \text{of } P_G \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \forall \lambda \in \Lambda_+, \text{ have line-bundle } \mathcal{L}_\lambda \subseteq P(V_\lambda), \ \text{irrep of h.w. } \lambda \\ \text{with v. b. quotient} \quad \mathcal{L}_{\lambda+\mu} \xrightarrow{\sim} \mathcal{L}_\lambda \otimes_A \mathcal{L}_\mu \\ \text{compatible w/ maps } P(V_{\lambda+\mu}) \longrightarrow P(V_\lambda) \otimes P(V_\mu) \end{matrix} \right\}$$

<u>Pf</u> Present $\bullet / B = T \backslash (G/N) / G$

Constructing maps $\mathrm{Spec}\, A \longrightarrow \overline{G/N}$ - base affine space.

$$\overline{G/N} = \mathrm{Spec}(\mathrm{Fun}(G)^N)$$

$$\mathrm{Fun}\, G = \bigoplus_{\lambda \in \Lambda_+} V_\lambda \otimes_{\mathbb{C}} V_\lambda^*$$

$$(\mathrm{Fun}\, G)^N = \bigoplus_{\lambda \in \Lambda_+} \mathbb{C}_\lambda \otimes V_\lambda^*, \quad \mathbb{C}_\lambda \text{ - spanned by h.w. vector}$$

Algebra structure: there is map $V_\lambda^* \otimes V_\mu^* \twoheadrightarrow V_{\lambda+\mu}^*$.

$$\mathrm{Map}(\mathrm{Spec}\, A, \overline{G/N}) = \left\{ \begin{array}{l} \forall \lambda \in \Lambda_+, \text{ a map } f_\lambda : \mathbb{C}_\lambda \otimes A \to V_\lambda \otimes A \\ \text{plus conditions for } \underbrace{\lambda+\mu, \text{etc}}_{\text{"Plucker identities.}} \end{array} \right\}$$

<u>Fact</u> $G/N \hookrightarrow \overline{G/N}$ open subscheme

$$\Rightarrow \mathrm{Map}(\mathrm{Spec} A, G/N) = \left\{ \begin{array}{l} \forall \lambda \in \Lambda_+, \\ f_\lambda \end{array} \middle| \text{ injective v.b. quotient} \right\}$$

$$\Rightarrow \mathrm{Map}(\mathrm{Spec} A, \underbrace{T \backslash G/N}_{= \cdot/B} /G) = \left\{ (P_G, \mathcal{L}_\lambda, f_\lambda)_{\lambda \in \Lambda_+} \middle| \begin{array}{l} f_\lambda : \mathcal{L}_\lambda \to P(V_\lambda) \text{ inj.} \\ \text{v.b. quotient \& satisfying} \\ \text{Plucker relations.} \end{array} \right\}$$

How to construct $\mathcal{L}_\lambda$ on $\mathrm{Spec}\, \mathfrak{z}$? (in terms of Plucker construction)

$\lambda^\vee : \mathbb{G}_m \to \check{H}$ $(H^\vee = T^\vee$

<u>Thm</u> (BD) Take $V_{\lambda^\vee}$ - irrep of $\check{G}$. The lowest eigenvalue of $L_0 = -t\frac{\partial}{\partial t}$ on $\Gamma_{\mathrm{crit}}(\mathrm{Gr}_G, \mathcal{F}_{V_{\lambda^\vee}})$ is $-\langle \lambda^\vee, \rho \rangle$.

This eigenspace is 1-dim.

<u>Rmks</u> Let $\mathbb{C}_{\lambda^\vee}$ be this eigenspace. Then $\mathbb{C}_{\lambda^\vee} \subseteq \Gamma(\mathrm{Gr}_G, \mathcal{F}_{V_{\lambda^\vee}})^{G(O)}$

$L_i := -t^{i+1}\frac{\partial}{\partial t}$. Then L_0 acts on $L_i \mathbb{C}_{\lambda^\vee}$ by $-\langle \check{\lambda}, \rho \rangle - i$.

$\Rightarrow$ For $i > 0$, this must be 0

$$\mathcal{L}_{\lambda^\vee} := \mathbb{C}_{\lambda^\vee} \otimes_{\mathbb{C}} \mathfrak{z} \hookrightarrow P(V_{\lambda^\vee}) = \Gamma(\mathrm{Gr}_G, \mathcal{F}_{V_{\lambda^\vee}})^{G(O)}.$$

<u>Claim</u> This satisfies necessary condition for $B^\vee$-reduction.

Rmk $\mathcal{L}_{\lambda^\vee}$ are Aut_* - invariant $\Rightarrow$ corresponding $\check{B}$ - reduction

$P_{B^\vee}$ is Aut_* -equivariant.

In summary, · $P_{G^\vee}$ - $G^\vee$-bundle on Spec $\mathfrak{z}$ Aut-eq.

· $P_{B^\vee}$ - $B^\vee$-reduction on Spec $\mathfrak{z}$ Aut_* -eq

Want: An Aut-eq. family of opers on Spec $\mathfrak{z} \times \mathbb{D}$.

Let S = affine scheme, Aut $\curvearrowright S$, $i: S \hookrightarrow S \times \mathbb{D}$.

Prop i) $i^*: \left\{ \begin{matrix} \text{Aut-eq. } G^\vee\text{-bundles} \\ \text{on } S \times \mathbb{D} \end{matrix} \right\} \xrightarrow{\sim} \left\{ \begin{matrix} \mathrm{Aut}_* \text{-eq } G^\vee\text{-bundles} \\ \text{on } S \end{matrix} \right\}$

ii) $i^*: \left\{ \begin{matrix} \text{Aut-eq } G^\vee\text{-bundles} \\ \text{on } S \times \mathbb{D} \text{ w/ connections} \\ \text{on } \mathbb{D} \end{matrix} \right\} \xrightarrow{\sim} \left\{ \begin{matrix} \text{Aut-eq. } G^\vee\text{-bundles} \\ \text{on } S \end{matrix} \right\}$

Pf 1) $\mathbb{D} = \mathrm{Aut}/\mathrm{Aut}_*$

$S \times \mathbb{D} / \mathrm{Aut} \cong (S \times \mathrm{Aut}/\mathrm{Aut}_*)/\mathrm{Aut} \cong S/\mathrm{Aut}_*$

2) $i: S \hookrightarrow S \times \mathbb{D}_{DR} \cong S$ Aut-equiv.

$\Rightarrow$ Get equivalence as desired. ∎

Using prop:

$\left(\begin{matrix} P_{G^\vee} \text{ on Spec } \mathfrak{z} \\ \text{Aut-eq} \end{matrix} \right) \rightsquigarrow \left(\begin{matrix} \text{family of } G^\vee\text{-local systems} \\ \text{Spec } \mathfrak{z} \times \mathbb{D}, \text{ Aut-eq.} \end{matrix} \right)$

$\left(\begin{matrix} P_{B^\vee} \text{ on Spec } \mathfrak{z} \\ \mathrm{Aut}_* \text{-eq} \end{matrix} \right) \rightsquigarrow \left(\begin{matrix} B^\vee \text{-reduction of} \\ \text{above on Spec } \mathfrak{z} \times \mathbb{D}, \text{ Aut-eq.} \end{matrix} \right)$

<u>Prop</u> Get family of $G^\vee$-opers on $\mathbb{D} \times \operatorname{Spec} \mathfrak{z}_\chi$

the condition: $H^\vee$-bundle induced by $B^\vee \to H^\vee$ has the form

$\rho(\omega_{\mathbb{D}})$ via $\rho: \mathbb{G}_m \to H^\vee$.
